

Basis of the Method

The collocation method is one of the methods of weighted residuals. These methods start by using a linear combination of trial functions to approximate the solution. Suppose we seek a solution of the equation:

$$u_{xx} + g(x, u, u_x) = 0 \quad (1)$$

for $0 < x < 1$, with $u(0) = u_0$ and $u(1) = u_1$. The solution is approximated by:

$$u \cong \tilde{u} = \sum_{i=1}^n a_i v_i(x) \quad (2)$$

Where the functions, v , are called trial functions or test functions. The trial functions are selected to meet the boundary conditions. The residual is formed by substituting the approximate solution into the differential equation:

$$R(x, \mathbf{a}) = \sum_{i=1}^n a_i v_{ixx} + g(x, \tilde{u}, \tilde{u}_x) \quad (3)$$

If the residual is zero for all x , an exact solution has been achieved. In general, this will not be possible, so the parameters, \mathbf{a} , are selected so the residual will be small in some sense. If the residual is small for all x , Eq. (2) will be a good approximation to the true solution. This goal is achieved by forcing the function to zero in a weighted average sense:

$$\int_0^1 w_i(x) R dx = 0, \quad \text{for } i=1, \dots, n \quad (4)$$

Different choices for the n weight functions produce different methods, some of the popular choices are:

1. $w_i = x^{(i-1)}$, Moments method
2. $w_i = v_i(x)$, Galerkin method
3. $w_i = \delta(x - x_i)$, Collocation method

The Moments method forces the residual to zero when weighted by increasingly higher members of a complete set of functions, so as n becomes large the residual must approach zero.

The Galerkin method weights the residual by the trial functions. Since the trial functions should be chosen from a complete set of functions, this method also guarantees the residual will approach zero for large n . Also, the Galerkin method is related to the Raleigh-Ritz method which is known to produce good results. The Raleigh-Ritz method requires that the solution obey a variational principal such as the minimization of energy. Problems amenable to this treatment are invariably linear. The Galerkin method produces identical approximations for these problems; however, the Galerkin method is a general method, which does not require linearity or a variational principal. Due to this relationship, the method is frequently called the Raleigh-Ritz-Galerkin method.

The collocation method sets the residual to zero at selected collocation points, so the weight functions may be viewed as dirac delta functions. Collocation is much simpler than the other methods, since integration is not required. However, the method can be dangerous if the collocation points are not carefully selected. For example, the collocation method can diverge for large n with equally spaced collocation points. We will show how

to select the collocation points to achieve accuracy similar to the Galerkin or moments methods while maintaining the simplicity of the collocation method.

Selection of Trial Functions

For most problems of practical interest, the general nature of the solution is known. A nice feature of the method of weighted residuals is that the trial functions can be selected to mimic the known characteristics of the solution. Trial function selection should also consider their suitability for calculations with a computer and also to make the method intuitive.

Polynomials are simple functions which are amenable to efficient computer calculations. Most descriptions of orthogonal collocation state that the trial functions are orthogonal polynomials. However, simple powers, $x^{(i-1)}$, are often used to develop the matrix operators, which work with nodal values, i.e. the values $\tilde{u}(x_i)$. A nodal representation is more intuitive than one using coefficients of orthogonal polynomials or simple powers. If the \mathbf{a} in Eq.(2) are the nodal values, $\tilde{u}(x_i)$, then the trial functions are not orthogonal polynomials, but rather the Lagrange interpolating polynomials:

$$v_i(x) = \ell_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(x - x_j)}{(x_i - x_j)} \quad (5)$$

With this choice of trial function, the approximate solution, Eq. (2) is:

$$u \cong \tilde{u} = \sum_{i=1}^n \tilde{u}_i \ell_i(x) \quad (6)$$

where $\tilde{u}_i = \tilde{u}(x_i)$.

There are many problems for which the solution is symmetric about $x = 0$. In this case Lagrange interpolating polynomials in x^2 are used for the trial functions:

$$v_i(x^2) = \ell_i(x^2) = \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(x^2 - x_j^2)}{(x_i^2 - x_j^2)} \quad (7)$$

Approximation of Integrals

When solving problems with the Galerkin method or Method of Moments, one rarely evaluates the integrals analytically. Instead, the integrals are usually approximated numerically using a quadrature formula of the form:

$$\int_0^1 f(x) x^\gamma dx = \sum_{i=1}^m W_i f(x_i) \quad (8)$$

The term x^γ is included, since we are interested in radial ($\gamma=1$) and spherical coordinates ($\gamma=2$) as well as normal planar geometry ($\gamma=0$). Considering the number of function evaluations required, the most efficient method for approximating integrals is Gaussian quadrature. It gives the exact integral for polynomials through degree $2m-1$. The Gaussian quadrature base points are the roots of Legendre orthogonal polynomials and all lie within the interval, $0 < x_i < 1$.

We will also find it useful to use quadrature formulae which include the endpoints, i.e. $x_1 = 0$ and $x_m = 1$. The most efficient method of this type is Lobatto quadrature. It gives the exact integral for polynomials through degree $2m-3$. Lobatto quadrature appears to be less efficient than Gaussian quadrature; however, if the function values at the endpoints are simple, e.g. if they are zero, two additional degrees of accuracy are achieved for little or no effort. The base points for Lobatto quadrature are the roots of Jacobi polynomials, and all lie within the interval.

Quadrature formulae can also be developed for symmetric problems. Gaussian quadrature for symmetric problems will integrate exactly polynomials of degree $2m-1$ in x^2 (or $4m-2$ in x). If the endpoint, $x_m = 1$, is included the Lobatto formula is exact for a polynomial of degree $2m-2$ in x^2 .

Appendix A describes methods for calculating the base points, \mathbf{x} , and weights, \mathbf{W} , for quadrature formulae of interest.